Second-harmonic resonant excitation in optical periodic structures with nonlinear anisotropic layers

A. A. Bulgakov,* S. A. Bulgakov,† and L. Vázquez
Departamento de Matemática Aplicada, Escuela Superior de Informática, Universidad Complutense, 28040 Madrid, Spain
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We propose a method to derive a system of ordinary differential equations (coupling equations) in order to analyze nonlinear wave processes inside lossless layered periodic media. We solve the problem associated with the interaction between the first- and second-harmonic waves inside a structure with anisotropic nonlinear dielectric layers. An arbitrary angle between the wave propagation direction and the structure interfaces is considered. The method takes into account both the nonlinear processes occurring at the slab interfaces and the nonlinear terms that depend differently on the field components. We derive analytically the phase matching conditions that provide the maximum interaction between the first- and second-harmonic waves. A physical explanation of the increase in efficiency of the wave coupling that takes place at the frequency near the passband edges is given. [S1063-651X(98)04111-7]

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I. INTRODUCTION

Studies of the nonlinear processes associated with the wave interaction inside inhomogeneous media (e.g., harmonics generation) show that the efficiency of frequency conversion increases in the vicinity of the passband edges [1–4]. The cause of such an increase is the low group velocity of the harmonic wave achieved at the frequency near the passband edge. The low group velocity results in the increase of a distance corresponding to the effective interaction of waves.

The electrodynamic properties of layered periodic structures have been intensively studied for more than 20 years [5–12]. In Ref. [5] Pozhar and Chernozatonski applied a mathematical model to show that the translation symmetry may considerably affect the processes of resonant excitation of waves; however, the fundamental physical properties of the phenomenon observed were not discussed. The same authors derived an estimation for the increase of the generation efficiency, namely, $(|R|M/2\pi)^2$, where $|R|$ is the reflection coefficient of one period and M is the number of the lattice periods. They [5] established that the conversion efficiency may be increased by a factor of about 500 times. Theoretical and experimental studies of the nonlinear resonant excitation of harmonics were carried out for liquid crystals [2,3,6]. A good correspondence between theory and experiments was verified.

Periodic structures appeared to be very effective for the efficient generation of harmonic waves, which may be used for practical purposes such as either frequency multiplication or conversion. Because of nonlinearity, the electromagnetic waves propagating in the structure with different wave numbers and frequencies may interact, i.e., exchange their energy with each other. Most studies of nonlinear media deal with so-called Kerr nonlinearity, which, at least for a one-dimensional periodic system, permits an analytical solution (see Ref. [11] and references therein). For instance, assuming that the Kerr nonlinear coefficient is small, Scalora *et al.* [13] solved the problem of optical pulse propagation through a one-dimensional dielectric (photonic) lattice.

To study the nonlinear processes inside finite and inhomogeneous structures a *three-wave interaction* (TWI) method was proposed [14–17]. This method does not make any restriction on a kind of nonlinearity applied. However, the smallness of the nonlinear terms is assumed. With the help of the TWI method, the system of differential equations in partial derivatives (i.e., Maxwell equations) can be reduced to a system of coupling ordinary differential equations for the slowly varying amplitudes of interacting waves.

The present work studies the nonlinear interaction of waves inside a two-dimensional layered structure that is periodic in one dimension and consists of alternating dielectric layers, namely, uniaxial anisotropic nonlinear slabs and isotropic linear dielectrics. For the optical frequency band, the width of the layers is about 0.3–3 μm. To simplify the problem, we consider a dielectric lattice belonging to the $6mm$-symmetry class: If the lattice axis coincides with the optical axis of the crystal then the anisotropic slab is homogeneous in the plane of layers (see the discussion in Sec. II A).

To our knowledge, the existing theoretical studies (those based on the TWI method for a periodic medium) deal only with wave propagation perpendicular to the slab interfaces. However, it is well known that eigenwaves in periodic media result from the interference processes inside the structure layers. Hence these natural modes are ''collective'' modes consisting of *elementary excitations* of layers. The elementary excitations are waves propagating along the slab interfaces. The fields of these waves satisfy the boundary conditions. The nonlinear processes associated with waves propagating obliquely with respect to the slab interfaces ap-

*Present address: Institute of Radiophysics and Electronics National Academy of Sciences of Ukraine, 12 Academic Proscura Street, 310085 Kharkov, Ukraine.

†Present address: Institute of Radio Astronomy, National Academy of Sciences of Ukraine, 4 Krasnoznamennaya Street, 310002 Kharkov, Ukraine.

pear to be more diverse and interesting than those nonlinear phenomena previously observed (see Refs. [1–3,6]).

The present work uses the TWI method and considers any value of the angle between the wave propagation direction and the structure interfaces. It should be noted that Refs. [1–3,5,6] study only the initial stage of the nonlinear process evolution. In contrast, we derive coupling equations that can be applied to a periodic medium and allow a complete analysis of the nonlinear process evolution. In this paper we derive analytic expressions for the phase matching conditions between the first- and second-harmonic waves and point out particularities of the TWI processes inside a periodic structure in comparison to those of a homogeneous medium.

In Sec. II we state the problem and describe briefly the three-wave interaction technique for periodic structures. This section discusses also the choice of the crystallographic symmetry and derives basic relations for the linear problem. Section III explains how to derive the nonlinear coupling equations. In Sec. IV we analyze the phase matching conditions between the first- and second-harmonic waves. Section V studies properties of the nonlinear interaction coefficient. Analytic and numerical solutions of the coupling equations are shown in Sec. VI. In Sec. VII we discuss the characteristics of nonlinear interactions in periodic media.

II. MATHEMATICAL MODEL AND PROBLEM

A. General system of equations

We study the nonlinear interaction between the first- and second-harmonic waves in a layered periodic lossless and dispersionless dielectric structure. We shall consider (see the discussion below) a uniaxial dielectric crystal belonging to the $6mm$-symmetry class. Such a symmetry reduces the problem configuration into a structure that is periodic in one direction and homogeneous in the plane of layers. Our method uses Green's formula [18] to derive a system of coupling equations. Our structure consists of alternating anisotropic and isotropic dielectric layers. Specifically, $\hat{\varepsilon}=(\varepsilon_{xx},\ \varepsilon_{xx},\ \varepsilon_{zz})$ is the dielectric permittivity of the first slab of width d_1, and ε_2 and d_2 are the dielectric permittivity and the width of the second layer, respectively. $d=d_1+d_2$ is the structure period. It is also assumed that the first layer is nonlinear.

The electromagnetic wave propagation inside each slab is described by Maxwell's equations, constitutive relations, and boundary conditions derived from the continuity of the tangential components of the electric and magnetic fields at the slab interfaces. Next one can choose a problem configuration so that the system of equations will be split in two independent polarizations. Specifically, such a separation of the equations is possible when the optical axis of the uniaxial dielectric coincides with the axis of the superlattice. In our problem the Z axis coincides with the optical axis of the crystal.

For an isotropic dielectric, all coefficients of the nonlinear polarization that may cause second-harmonic generation equal zero [14]. Therefore, we shall consider a second-order nonlinearity (see Refs. [14,19]) supposing that one layer in a unit cell contains a uniaxial dielectric medium with nonzero third-order coefficients (χ_{ikl}) of the nonlinear polarization. By expanding the polarization vector $\vec{P}$ into a series of electrical field components, for the nonlinear components P_i^{nl} ($i=x,y,z$) of $\vec{P}$ one has the expression (see Ref. [19], Chap. 12)

$$\begin{pmatrix} P_x^{nl} \\ P_y^{nl} \\ P_z^{nl} \end{pmatrix} = \begin{pmatrix} \chi_{xxx} & \overline{\chi_{xyy}} & \chi_{xzz} & \overline{\chi_{xyz}} & \chi_{xxz} & \overline{\chi_{xxy}} \\ \chi_{yxx} & \overline{\chi_{yyy}} & \chi_{yzz} & \overline{\chi_{yyz}} & \chi_{yxz} & \overline{\chi_{yxy}} \\ \chi_{zxx} & \overline{\chi_{zyy}} & \chi_{zzz} & \overline{\chi_{zyz}} & \chi_{zxz} & \overline{\chi_{zxy}} \end{pmatrix} \times \begin{pmatrix} E_x^2 \\ E_y^2 \\ E_z^2 \\ E_zE_y \\ E_zE_x \\ E_xE_y \end{pmatrix}. \tag{1}$$

One can verify that to derive a system of equations for the fields of two independent polarizations, the following components of χ_{ijk} must be zero: $\chi_{yxx}=\chi_{yzz}=\chi_{yxz}=0$. Those elements of χ_{ijk}, denoted in Eq. (1) as χ_{ijk}, may have arbitrary values. For instance, the following crystalline substances possess the necessary susceptibility tensor: CdS, α-ZnS, ZnO, BeO, SiC, AgI, and CaAs are of hexagonal symmetry; $LiNbO_3$, $LiTaO_3$, Ag_3SbS_3, α-quartz, HgS, and Se are of trigonal symmetry; and $BaTiO_3$, $PbTiO_3$, and $SbN(Sr_{0.5}Ba_{0.5}Nb_2O_6)$ are tetragonal lattices.

Let us consider the optical axis, namely, the Z axis, perpendicular to the slab interfaces. Next, the system of Maxwell's equations will be split into two independent polarizations: E_x,E_z,H_y and E_y,H_x,H_z. We shall consider an anisotropic dielectric belonging to the $6mm$-symmetry group [19] with the following tensor of nonlinear susceptibility:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \chi_{xxz} & 0 \\ 0 & 0 & 0 & \chi_{xxz} & 0 & 0 \\ \chi_{zxx} & \chi_{zyy} & \chi_{zzz} & 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

Taking into account the nonlinear terms, let us write Maxwell's equations inside the first slab for polarization E_x,E_z,H_y as

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{1}{c}\frac{\partial H_y}{\partial t},$$

$$-\frac{\partial H_y}{\partial z}=\frac{\varepsilon_{xx}}{c}\frac{\partial E_x}{\partial t}+\frac{4\pi}{c}\chi_{xxz}\frac{\partial}{\partial t}(E_xE_z), \tag{3}$$

$$\frac{\partial H_y}{\partial x}=\frac{\varepsilon_{zz}}{c}\frac{\partial E_z}{\partial t}+\frac{4\pi}{c}\chi_{zxx}\frac{\partial}{\partial t}(E_xE_x)+\frac{4\pi}{c}\chi_{zzz}\frac{\partial}{\partial t}(E_zE_z).$$

One can see that due to the choice of $6mm$-symmetry class, our problem is homogeneous in the plane of layers. Thus one can set $\partial/\partial y=0$. Equations for the field inside the second slab are similar to Eqs. (3), but without the nonlinear terms and with ε_2 instead of $\varepsilon_{zz},\varepsilon_{xx}$.

B. Linear solution

The linearized system of equations (3) has been studied in the literature [7,8]. Applying the technique of the transfer matrix method as it is described in Ref. [8] by using the Floquet theorem and taking into account the boundary conditions at $z=d_1$, we derive the transfer matrix as

$$\begin{pmatrix} H_y(0) \\ E_x(0) \end{pmatrix} = \hat{m} \begin{pmatrix} H_y(d) \\ E_x(d) \end{pmatrix}; \tag{4}$$

$$m_{11}=\cos k_{z1}d_1 \cos k_{z2}d_2 - \frac{k_{z2}\varepsilon_{xx}}{k_{z1}\varepsilon_2}\sin k_{z1}d_1 \sin k_{z2}d_2,$$

$$m_{22}=\cos k_{z1}d_1 \cos k_{z2}d_2 - \frac{k_{z1}\varepsilon_2}{k_{z2}\varepsilon_{xx}}\sin k_{z1}d_1 \sin k_{z2}d_2,$$

$$m_{12}=-i\frac{\omega}{c}\left(\frac{\varepsilon_2}{k_{z2}}\cos k_{z1}d_1 \sin k_{z2}d_2 + \frac{\varepsilon_{xx}}{k_{z1}}\sin k_{z1}d_1 \cos k_{z2}d_2\right),$$

$$m_{21}=-i\frac{c}{\omega}\left(\frac{k_{z1}}{\varepsilon_{xx}}\sin k_{z1}d_1 \cos k_{z2}d_2 + \frac{k_{z2}}{\varepsilon_2}\cos k_{z1}d_1 \sin k_{z2}d_2\right). \tag{5}$$

Here

$$k_{z1}=\sqrt{\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\left(\frac{\omega^2}{c^2}\varepsilon_{zz}-k_x^2\right)}, \quad k_{z2}=\sqrt{\frac{\omega^2}{c^2}\varepsilon_2-k_x^2} \tag{6}$$

are the transversal wave-vector components inside the first and second slabs, respectively. k_x is the wave number along the X axis directed parallel to the slab interfaces. The transfer matrix (5) appearing in Eq. (4) differs from that of Ref. [8], namely, it takes into account the anisotropy of the first layer of the unit cell.

Making use of the Floquet theorem, namely, $H_y(d) = H_y(0)\exp(i\bar{k}d)$, $E_x(d)=E_x(0)\exp(i\bar{k}d)$, one derives the dispersion relation for a periodic medium as

$$\cos \bar{k}d = \frac{m_{11}+m_{22}}{2}. \tag{7}$$

Here $\bar{k}$ is the Bloch wave vector. The dispersion relation (7) connects the values $\omega,\bar{k},k_x$.

To proceed, we need the field expressions

$$E_{x1}=C\left[\frac{ick_{z1}}{\omega\varepsilon_{xx}}\sin k_{z1}z + A\cos k_{z1}z\right],$$
$$E_{z1}=-C\frac{k_x c}{\omega\varepsilon_{zz}}\left[\cos k_{z1}z + A\frac{i\omega\varepsilon_{xx}}{k_{z1}c}\sin k_{z1}z\right] \tag{8}$$

for $Nd \leqslant z < d_1+Nd$ and

$$E_{x2}=C\left[B_1\frac{ick_{z2}}{\omega\varepsilon_2}\sin k_{z2}z + B_2\cos k_{z2}z\right],$$

$$E_{z2}=-C\frac{k_x c}{\omega\varepsilon_2}\left[B_1\cos k_{z2}z + B_2\frac{i\omega\varepsilon_2}{k_{z2}c}\sin k_{z2}z\right]$$

for $d_1+Nd \leqslant z < (N+1)d$. Here A,B_1,B_2,C are constant and can be determined from the boundary conditions and Floquet theorem. N is an integer. It is also supposed that the field dependence on time t and the space coordinate x is $\exp(-i\omega t+ik_x x)$. Next one derives the expressions for B_1,B_2,A, namely,

$$B_1=\cos k_{z1}d_1\cos k_{z2}d_1 + \frac{k_{z1}\varepsilon_2}{k_{z2}\varepsilon_{xx}}\sin k_{z1}d_1 \sin k_{z2}d_1 + \left(-i\frac{\omega}{c}\right)A\left(\frac{\varepsilon_2}{k_{z2}}\cos k_{z1}d_1\sin k_{z2}d_1 - \frac{\varepsilon_{xx}}{k_{z1}}\sin k_{z1}d_1\cos k_{z2}d_1\right),$$

$$B_2=\left(-i\frac{c}{\omega}\right)\left(\frac{k_{z2}}{\varepsilon_2}\cos k_{z1}d_1\sin k_{z2}d_1 - \frac{k_{z1}}{\varepsilon_{xx}}\sin k_{z1}d_1\cos k_{z2}d_1\right) + A\left(\cos k_{z1}d_1\cos k_{z2}d_1 + \frac{k_{z2}\varepsilon_{xx}}{k_{z1}\varepsilon_2}\sin k_{z1}d_1\sin k_{z2}d_1\right), \tag{9}$$

$$A=\frac{m_{22}-\exp(i\bar{k}d)}{m_{12}}.$$

Making use of the Green's formula [18], we derive in the next section the *coupling equations* that describe the interaction between the first- and second-harmonic waves.

III. COUPLING EQUATIONS

Next we solve the nonlinear problem taking into account the smallness of the nonlinear terms in Eqs. (3). We shall look for solution of Eqs. (3) in the form of a sum over spatial harmonics with different longitudinal wave vector k_x (cf. [20]):

$$\vec{E}=\sum_{k_x=-\infty}^{\infty} C(z)[\vec{e}(z)+\vec{e}^{(ad)}(z)]\exp(-i\omega t+ik_x x),$$
$$H_y=\sum_{k_x=-\infty}^{\infty} C(z)[h_y(z)+h_y^{(ad)}(z)]\exp(-i\omega t+ik_x x). \tag{10}$$

Here $C(z)$ is the slowly varying amplitude [21] of the kth wave (that is, the wave with the longitudinal wave number k_x and frequency ω):

$$\frac{d\ln C}{dz} \ll \bar{k},k_{z1,2}. \tag{11}$$

In Eqs. (10) both values of $C(z)$ and the fields in square brackets are different for different kth waves (for simplicity, we do not assign any index, say, index k, for each of the values). Also the frequency ω, k_x, and the transversal wave-number component $\bar{k}$ are coupled by the dispersion relation (7).

It should be noticed here that k_x value is unambiguously related to the angle between the wave propagation direction and the structure interfaces. Thus, in Eqs. (10) *any* possible directions of the wave propagation are taken into account.

The fields $\vec{e}(z)$ and $h_y(z)$ are solutions of the homogeneous linear system of equations, i.e., of Eqs. (3) without the nonlinear terms (cf. [15,16]). Inside the structure layers, these fields contain the fast factors such as $\exp(\pm ik_{z1,2}z)$ in their dependences on z. Additional fields $\vec{e}^{(ad)}$ and $h_y^{(ad)}$ are partial solutions of the inhomogeneous system of equations, namely, of Eqs. (3) with the nonlinear terms taken as the right-hand side of the system. Also notice that $\vec{e}^{(ad)}, h_y^{(ad)}$ appear to be of the order of the nonlinear terms.

A solution in the form of Eqs. (10) differs from the traditional technique of analysis of the TWI processes [15,16]. Specifically, in Eqs. (10) not only C but also the components $\vec{e}(z), h_y$ depend on z. The latter stems from the fact that our structure is inhomogeneous in the Z direction.

Next we average the system of Maxwell equations over "fast" field variations in time and space, i.e., over dependences such as $\exp(\pm ik_x x - i\omega t)$ and the one associated with the field variation due to periodicity along the Z axis. Our goal is to extract a slow dependence of the electromagnetic field amplitude C on z and finally derive a system of equations for C. To this end, we substitute the fields (10) into Eqs. (3), multiply the right- and left-hand sides of Eqs. (3) by $\exp(i\omega t - ik_x x)$, and integrate the equations over time $(-\infty < t < +\infty)$ and over x $(-\infty < x < +\infty)$. Finally, the system of Maxwell's equations (3) takes the form

$$\hat{L}f = \hat{\mathcal{H}}(f,f) - \hat{L}f^{(ad)}. \tag{12}$$

$\hat{L}$ and $\hat{\mathcal{H}}$ are linear and nonlinear operators, respectively. f is a solution of the system of equations $\hat{L}f = 0$. Actually, both f and $f^{(ad)}$ are the column vectors consisting of the field components of the ith slab inside a period:

$$f_i = \begin{pmatrix} h_{yi} \\ e_{xi} \\ e_{zi} \end{pmatrix}, \quad f_i^{(ad)} = \begin{pmatrix} h_{yi}^{(ad)} \\ e_{xi}^{(ad)} \\ e_{zi}^{(ad)} \end{pmatrix}, \quad i = 1,2.$$

Observe that due to the smallness of nonlinearity, $\hat{\mathcal{H}}(f,f)$ contains no additional fields $f^{(ad)}$.

Equation (12) is the inhomogeneous system of linear differential equations. $\hat{L}$ and $\hat{\mathcal{H}}$ are defined inside the slabs as follows: For the first layer $(Nd \leqslant z < d_1 + Nd)$

$$\hat{L}(f_1 + f_1^{(ad)}) = \begin{pmatrix} \frac{-i\omega}{c} & \frac{\partial}{\partial z} & -ik_x \\ -\frac{\partial}{\partial z} & \frac{i\omega}{c}\varepsilon_{xx} & 0 \\ ik_x & 0 & \frac{i\omega}{c}\varepsilon_{zz} \end{pmatrix} \begin{pmatrix} C(h_{y1} + h_{y1}^{(ad)}) \\ C(e_{x1} + e_{x1}^{(ad)}) \\ C(e_{z1} + e_{z1}^{(ad)}) \end{pmatrix},$$

$$\hat{\mathcal{H}}(f_1, f_1) = \begin{pmatrix} -e_{x1}\frac{dC}{dz} \\ h_{y1}\frac{dC}{dz} + \frac{4\pi}{c}\chi_{xxz}C'C''[-i(\omega' + \omega'')] \times \frac{1}{2}(e'_{x1}e''_{z1} + e'_{z1}e''_{x1}) \\ \frac{4\pi}{c}C'C''[-i(\omega' + \omega'')] \times (\chi_{zxx}e'_{x1}e''_{x1} + \chi_{zzz}e'_{z1}e''_{z1}) \end{pmatrix} \tag{13}$$

and for the second layer $[d_1 + Nd \leqslant z < (N+1)d]$

$$\hat{L}(f_2 + f_2^{(ad)}) = \begin{pmatrix} \frac{-i\omega}{c} & \frac{\partial}{\partial z} & -ik_x \\ -\frac{\partial}{\partial z} & \frac{i\omega}{c}\varepsilon_2 & 0 \\ ik_x & 0 & \frac{i\omega}{c}\varepsilon_2 \end{pmatrix} \begin{pmatrix} C(h_{y2} + h_{y2}^{(ad)}) \\ C(e_{x2} + e_{x2}^{(ad)}) \\ C(e_{z2} + e_{z2}^{(ad)}) \end{pmatrix}, \quad \hat{\mathcal{H}}(f_2, f_2) = \begin{pmatrix} -e_{x2}\frac{dC}{dz} \\ h_{y2}\frac{dC}{dz} \\ 0 \end{pmatrix}. \tag{14}$$

It should be noted that, despite the second slab consisting of a linear dielectric only, the nonlinear terms $\hat{\mathcal{H}}(f,f)$ appear in the field equations for both the first and second layers [Eqs. (13) and (14), respectively]. Observe in Eqs. (12)–(14) that after integration over t and x, the linear part of Eq. (12), $\hat{L}f$, contains only parameters of the kth wave (i.e., the wave with the longitudinal wave number k_x). The nonlinear terms $\hat{\mathcal{H}}(f,f)$ couple the fields of the k'th and k''th waves, i.e., those waves associated with the longitudinal wave numbers k_x' and k_x'' (the parameters of these waves are marked by one and two primes), respectively. In general, all values of k_x, k_x', and k_x'' are different. Let us also remark that the nonlinear terms in Eqs. (12)–(14) contain the product $C'C''$ of the slow varying amplitudes C' and C'' of the k'th and k''th waves, correspondingly.

Next we proceed with averaging of Eq. (12) over fast field variations along the Z axis. This will be done with the aid of the Green's formula [18]. Keeping the notations we used in Eqs. (12)–(14), this formula can be written as

$$\int_a^b [\tilde{f}^*(\hat{L}f) - (\hat{\tilde{L}}\tilde{f})^* f]dz = ff^*|_a^b. \tag{15}$$

Here $\hat{\tilde{L}}$ is the transpose of $\hat{L}$, i.e., the transpose of the square matrix consisting of operators of the linearized system of equations, and $\tilde{f}$ is a solution of $\hat{\tilde{L}}$, $\tilde{f}=0$. An asterisk denotes complex conjugation. $\tilde{f}^*(\hat{L}f)$ and $(\hat{\tilde{L}}\tilde{f})^*f$ denote scalar products. It is also known that, in homogeneous media, $\hat{L}$ and $\hat{\tilde{L}}$ are algebraic operators [15].

The transpose $\hat{\tilde{L}}$ of the operator $\hat{L}$ is obtained using a standard procedure, namely, we exchange rows and columns in $\hat{L}$, substitute $\partial/\partial z$ by $-\partial/\partial z$, and apply complex conjugation. It turns out that each element of $\hat{\tilde{L}}$ is equal to the corresponding term of $\hat{L}$ taken with the opposite sign. Thus solutions of the initial and corresponding transposed system coincide. An important meaning of the Green's formula consists in the following: Eigenfunctions of the transposed linear differential operator are orthogonal to the right-hand side of the inhomogeneous system of linear differential equations, namely, eigenfunctions of $\hat{\tilde{L}}$ are orthogonal to the operator $\hat{\mathcal{H}}$.

Making use of Eq. (15), we find that the integration of the linear operators $\hat{L}$ and $\hat{\tilde{L}}$ yields the difference between the corresponding field components [Eq. (8)] at the structure boundaries. Both the main fields (8) and the additional fields $\vec{e}^{(ad)}, h_y^{(ad)}$ satisfy the boundary conditions. Therefore, the integrals of all linear terms are zeros.

Next we integrate the nonlinear terms $\hat{\mathcal{H}}(f,f)$. Let us represent the integral over z $(-\infty<z<\infty)$ as a sum of integrals associated with the structure layers:

$$\int_{-\infty}^{\infty} = \lim_{\delta_i\to 0, i=0,\pm 1,} \dots \left\{ \dots + \int_{-\delta_0}^{\delta_0} + \int_{\delta_0}^{-\delta_1+d_1} + \int_{-\delta_1+d_1}^{\delta_1+d_1} + \int_{\delta_1+d_1}^{-\delta_2+d} + \cdots \right\}.$$

Here we highlight the regions of widths $2\delta_i$ and $\delta_i+\delta_{i+1}$ in the vicinity of each boundary. With the help of the Floquet theorem, we express the fields in the integrals associated with layers in terms of the field of the first period of the structure. The following sum appears:

$$\int_{-\infty}^{\infty} = \lim_{\delta_0,\delta_1,\delta_2\to 0} \left(\int_{\delta_0}^{-\delta_1+d_1} + \int_{\delta_1+d_1}^{-\delta_2+d} \right) \times \sum_{N,N',N''} \exp\{i[(\overline{k'}+\overline{k''}-\overline{k})d + 2\pi(N'+N''-N)]\}.$$

N,N',N'' denote the period numbers associated with the kth, k'th, and k''th interacting waves, respectively, and

$$\sum_{N,N',N''} \exp\{i[(\overline{k'}+\overline{k''}-\overline{k})d + 2\pi(N'+N''-N)]\} = \lim_{M\to\infty} \frac{1-\exp\{iM[(\overline{k'}+\overline{k''}-\overline{k})d]\}}{1-\exp[i(\overline{k'}+\overline{k''}-\overline{k})d]} = \delta((\overline{k'}+\overline{k''}-\overline{k})d + 2\pi L), \quad L=0,\pm 1,\dots .$$

Finally, we obtain [22]

$$\frac{dC}{dz}\left\{ \int_0^{d_1} |h_{y1}e_{x1}|dz + \int_{d_1}^{d} |h_{y2}e_{x2}|dz \right\} = -i\frac{4\pi}{c} \sum_{k_x',k_x''} \left\{ \int_0^{d_1} \left[\frac{1}{2}\chi_{xxz}(e_{x1}^* e_{x1}' e_{z1}'' + e_{x1}^* e_{z1}' e_{x1}'') + \chi_{zxx} e_{z1}^* e_{x1}' e_{x1}'' + \chi_{zzz} e_{z1}^* e_{z1}' e_{z2}'' \right] \times dz\, \delta((\overline{k'}+\overline{k''}-\overline{k})d + 2\pi L) \exp[-i(\omega'+\omega''-\omega)t + i(k_x'+k_x''-k_x)x] \right\} C'C''. \tag{16}$$

If the following phase matching conditions are fulfilled, namely,

$$\omega'+\omega''-\omega=0, \tag{17a}$$

$$k_x'+k_x''-k_x=0, \tag{17b}$$

$$\overline{k'}+\overline{k''}-\overline{k}+\frac{2\pi L}{d}=0, \tag{17c}$$

then the equation for the amplitude C takes the form [16]

$$\frac{dC}{dz} = W_{k,k',k''}C'C'',$$

$$W_{k,k',k''}=-i\frac{4\pi}{c}\frac{\omega'+\omega''}{S_z}\times\int_0^{d_1}dz\left[\frac{\chi_{xxz}}{2}(e^*_{x1}e'_{x1}e''_{z1}+e^*_{x1}e'_{z1}e''_{x1})+\chi_{zxx}e^*_{z1}e'_{x1}e''_{x1}+\chi_{zzz}e^*_{z1}e'_{z1}e''_{z2}\right]. \quad (18)$$

Here S_z denotes the expression in large curly brackets after dC/dz on the left-hand side of Eq. (16). S_z is the energy flow of the kth wave in the Z direction. Thus $W_{k,k',k''}$ is a ratio of the energy flow along the Z axis (this flow appears due to the nonlinear interaction of waves) to the energy flux of the kth wave itself. The inequality (11) means $W_{k,k',k''}\ll 1$.

Equation (18) is written for the kth wave. Equations for the k'th and k''th waves may be derived by permutation of indices in Eq. (18). Then a system of equations for the three interacting waves will be derived.

It is necessary here to highlight the particularities of the nonlinear interaction in a periodic structure in comparison to the TWI processes in a homogeneous infinite medium.

(i) The dispersion relations for the three interacting waves contain Bloch wave vectors $\overline{k},\overline{k'},\overline{k''}$ instead of the transversal wave number k_z, namely, $\omega(\overline{k},k_x),\omega'(\overline{k'},k'_x)$, $\omega''(\overline{k''},k''_x)$.

(ii) Despite the first two laws of the energy conservation, namely, Eqs. (17a) and (17b), not differing from those of homogeneous media, the third relation (17c) contains the term $2\pi L/d$ associated with the structure periodicity. The latter means that umklapp phenomena (see Ref. [6]) exist in this periodic medium. The necessity of this third law was discussed from a physical viewpoint in Ref. [17].

(iii) The third particularity of the nonlinear interaction in a periodic structure is associated with the shape of the coefficient of the nonlinear interaction $W_{k,k',k''}$ [see Eq. (18)]. This coefficient is discussed in Sec. IV, where the interaction between the first- and second-harmonic waves is considered.

In Eqs. (10) [see also Eq. (18)], the slowly varying amplitude C is assumed to be dependent on z only. We highlight here that the Z axis should not be confused with the direction associated with the maximum nonlinear interaction between the harmonic waves. As is known, in general, a maximum nonlinear interaction (such as the amplification of the second-harmonic wave) occurs in a certain direction that has nonzero components on both the X and Z axes (see Ref. [4]). Equation (18) implies that the phase matching conditions (17) are satisfied. Thus those values of the longitudinal and transversal components of the wave vector that satisfy Eqs. (17b) and (17c), respectively, will determine the direction of the maximum nonlinear interaction between the harmonics. Below, in Sec. VI, we analyze the dependences of slowly varying amplitudes C,C' on z in order to determine the spatial distance of the complete energy exchange between the first- and second-harmonic waves.

IV. ANALYSIS OF PHASE MATCHING CONDITIONS

We shall consider the interaction of the k'th wave with given frequency ω' and longitudinal wave number k' with its second harmonic, say, the kth wave with frequency $2\omega'$ and the wave-number component $2k'_x$. Making use of the TWI method, we assume that the k'th and k''th waves are the same wave: $\omega''\equiv\omega',k''_x\equiv k'_x$, and $\overline{k''}\equiv\overline{k'}$. Thus the resonant conditions (17) take the forms

$$\omega=2\omega',\quad k_x=2k'_x,\quad \overline{k}=2\overline{k'}+2\pi L/d,\quad L=0,\pm1,\pm2,\ldots. \quad (19)$$

Then the system of equations for slowly varying amplitudes of the k'th wave and its second harmonic, i.e., the kth wave (amplitudes C' and C, respectively), takes the form

$$dC/dz=W_{2k',k'}C'^2,$$
$$dC'/dz=W_{k',2k'}CC'^*. \quad (20)$$

The matrix elements $W_{2k',k'},W_{k',2k'}$ are determined by Eq. (18) where values with one and two primes are equal each other. The values $\omega,k_x,\overline{k},\omega',k'_x,\overline{k'}$ satisfy the dispersion relations

$$\cos 2\overline{k'}d=\cos 2k'_{z1}d_1\cos 2k'_{z2}d_2-\frac{1}{2}\left(\frac{k'_{z2}\varepsilon_{xx}}{k'_{z1}\varepsilon_2}+\frac{k'_{z1}\varepsilon_2}{k'_{z2}\varepsilon_{xx}}\right)\sin 2k'_{z1}d_1\sin 2k'_{z2}d_2, \quad (21a)$$

$$\cos\overline{k'}d=\cos k'_{z1}d_1\cos k'_{z2}d_2-\frac{1}{2}\left(\frac{k'_{z2}\varepsilon_{xx}}{k'_{z1}\varepsilon_2}+\frac{k'_{z1}\varepsilon_2}{k'_{z2}\varepsilon_{xx}}\right)\sin k'_{z1}d_1\sin k'_{z2}d_2. \quad (21b)$$

Equations (21) take into account both relations (19) and $k_{z1,2}=2k'_{z1,2}$. The latter equality is valid in dispersionless dielectric media [to verify this one should substitute $k_x=2k'_x$ and $\omega=2\omega'$ into Eqs. (6)].

The five equations (19) and (21) contain six unknown values $\omega,\omega',k_x,k'_x,\overline{k},\overline{k'}$. Therefore, one of these values must be determined, e.g., the first-harmonic frequency ω' (or wave number k'_x). Then the functions $\overline{k'}(\omega')$ and $k'_x(\omega')$ can be determined with the help of Eqs. (21).

Equations (21) possess the exact solution

$$\cos\overline{k'}d=\cos(k'_{z1}d_1\pm k'_{z2}d_2). \quad (22)$$

The solution (22) is valid inside the passbands where the absolute value of the right-hand side of Eq. (7) is less than 1. Substituting Eq. (22) into Eq. (21b), one derives the relations for the function $k'_x(\omega')$ [or $\omega'(k'_x)$]:

$$\frac{(\varepsilon_{xx}k'_{z2}\pm\varepsilon_2k'_{z1})^2}{\varepsilon_{xx}\varepsilon_2}\frac{\sin k'_{z1}d_1}{k'_{z1}}\frac{\sin k'_{z2}d_2}{k'_{z2}}=0. \quad (23)$$

Thus the phase matching conditions (17) are fulfilled for the three cases

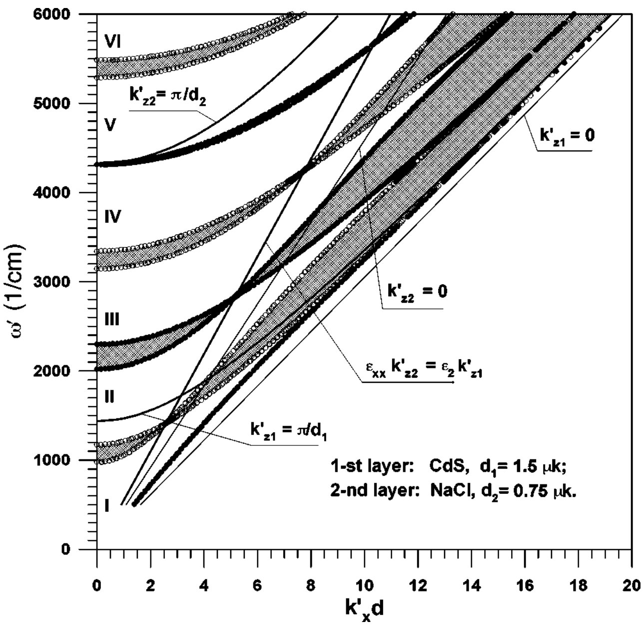


FIG. 1. Zone spectrum of the dielectric lattice. Frequency ω is plotted versus dimensionless longitudinal wave number $k_x d$. Stop bands are marked by filled pattern: light lines denote $k_{z1}=0$ (lower straight thin line) and $k_{z2}=0$ (upper straight thin line). The solutions of Eqs. (24) (thick curves) are $k'_{z1}=\pi/d_1$ [Eq. (24b)], zone II, and $k'_{z2}=\pi/d_2$ [Eq. (24c)], zone V. The thick straight line corresponds to $\varepsilon_{xx}k'_{z2}=\varepsilon_2 k'_{z1}$ [Eq. (24a)].

$$\varepsilon_{xx}k'_{z2}-\varepsilon_2 k'_{z1}=0, \quad k'_x=\frac{\omega'}{c}\sqrt{\frac{\varepsilon_2\varepsilon_{zz}(\varepsilon_{xx}-\varepsilon_2)}{\varepsilon_{xx}\varepsilon_{zz}-\varepsilon_2^2}}; \tag{24a}$$

$$k'_{z1}d_1=m_1\pi, \quad k'_x=\sqrt{\varepsilon_{zz}\left(\frac{\omega'^2}{c^2}-\frac{m_1^2\pi^2}{\varepsilon_{xx}d_1^2}\right)},$$

$$m_1=\pm 1,\pm 2,\ldots \tag{24b}$$

$$k'_{z2}d_2=m_2\pi, \quad k'_x=\sqrt{\frac{\omega'^2}{c^2}\varepsilon_2-\frac{m_2^2\pi^2}{d_2^2}},$$

$$m_2=\pm 1,\pm 2,\ldots. \tag{24c}$$

Equations (24) have a simple physical meaning. To analyze these relations we consider in Fig. 1 the passbands and stop bands of the lattice that consists of nonlinear dielectric CdS as the first layer inside the unit cell [$d_1=1.5$ μm, $\varepsilon_{xx}=5.382$, $\varepsilon_{zz}=5.457$, $\chi_{xxz}=210\times 10^{-9}$ CGSE (cantimeter-gramm-second-electrical system) units, $\chi_{zxx}=192\times 10^{-9}$ CGSE units, $\chi_{zzz}=378\times 10^{-9}$ CGSE units], and a homogeneous dielectric NaCl as the second slab ($d_2=0.75$ μm and $\varepsilon_2=2.38$). Filled circuits mark the band boundaries where $\cos \overline{k'}d=1$. Circumferences correspond to the curves with $\cos \overline{k'}d=-1$. Stop bands are marked by filled pattern. Each passband is numbered by a roman figure. The first passband (number I) starts from low frequencies ($\omega'\to 0$) and small wave numbers ($k'_x d\to 0$). With the increase of frequency, this passband becomes narrow and both of its boundaries approach an asymptotic line $\omega'=k'_x c/\varepsilon_{zz}$ ($k'_{z1}=0$) (thin line in Fig. 1). All the passbands touch each other in points that divide the corresponding stop band into two parts. It should be noted that the second asymptote $k'_{z2}=0$ (dashed line in Fig. 1) is tangential to the curves $\cos \overline{k'}d=\pm 1$ at the inflection points.

The thick straight line in Fig. 1 corresponds to Eq. (24a). This line is the dispersion curve $\omega=k_x c/\sqrt{\varepsilon_{eff}}$ of a light wave propagating in a homogeneous medium with an effective dielectric permittivity

$$\varepsilon_{eff}=\frac{\varepsilon_2\varepsilon_{zz}(\varepsilon_{xx}-\varepsilon_2)}{\varepsilon_{xx}\varepsilon_{zz}-\varepsilon_2^2}. \tag{25}$$

We now analyze the field expressions (8) and (9) for the case when the relation (25) is satisfied. Substituting Eq. (24a), namely, $k'_{z1}/\varepsilon_{xx}=k'_{z2}/\varepsilon_2$, into Eqs. (8) and (9), one obtains

$$E'_{x1}=C'\alpha' e^{ik'_{z1}z}, \quad 0\leqslant z<d_1$$

$$E'_{x2}=C'\alpha' e^{i(k'_{z1}-k'_{z2})d_1}e^{ik'_{z2}z}, \quad d_1\leqslant z<d$$

$$\alpha'=\sqrt{\frac{\varepsilon_{xx}\varepsilon_{zz}-\varepsilon_2^2-\varepsilon_{xx}^2+\varepsilon_2\varepsilon_{xx}}{\varepsilon_2(\varepsilon_{xx}\varepsilon_{zz}-\varepsilon_2^2)}}. \tag{26}$$

Expressions for the field of the second-harmonic wave are analogous. Equations (26) show that the traveling wave regime is set in the structure: There is no reflection from the boundaries. It should be noticed that for a homogeneous medium, the relation $\varepsilon_{xx}k'_{z2}=\varepsilon_2 k'_{z1}$ becomes identical. Therefore, this type of resonant interaction is analogous to those TWI phenomena that take place in homogeneous media with the effective dielectric permittivity (25). However, as it is seen in Fig. 1, the straight line [Eq. (24a)] crosses the stop bands where the relation $\varepsilon_{xx}k'_{z2}=\varepsilon_2 k'_{z1}$ is not valid. Therefore, in the vicinity of the band boundaries, the nonlinear interaction of waves cannot be studied with the help of Eq. (18): This equation is inapplicable inside the stop band regions.

Equations (24b) and (24c) are the Bragg resonance conditions for the first and second layers, namely, $k'_{z1}=m_1\pi/d_1$ and $k'_{z2}=m_2\pi/d_2$, respectively. The thick curves in Fig. 1 correspond to these equations (24b) and (24c). The curve $k'_{z1}=\pi/d_1$ starts in the second passband and, with the increase of $k'_x d$, crosses the stop band and approaches an asymptotic line $k'_{z1}=0$. Thus one should expect particularities of the nonlinear interaction in the vicinity of the boundary of the second passband. Only part of the curve for Eq. (24c) is depicted in Fig. 1. The slope of this curve is steeper than that of the line given by Eq. (24b). This is due to the choice of the values $\varepsilon_2>\varepsilon_{zz}$. Numerical calculations show that the curve for Eq. (24c) crosses a point belonging to both passband boundaries. Therefore, in contrast to Eq. (24a), the phase matching conditions given by Eqs. (24b) and (24c) are fulfilled along the curves that cross the boundary between the passbands and stop bands only one time.

In the cases of Eqs. (24b) and (24c), the fields of harmonic waves possess the following interesting particularity.

For the first harmonic wave, making use of the condition $k'_{z1}d_1 = l\pi$ $(l = 1,2,\dots)$, from Eqs. (8) and (9) one obtains

$$E'_{x1} = -C'\frac{k'_{z2}c}{\omega'\varepsilon_2}\left(\cos k'_{z1}z - i\frac{k'_{z1}\varepsilon_2}{k'_{z2}\varepsilon_{xx}}\sin k'_{z1}z\right), \quad 0 \leq z < d_1$$

$$E'_{x2} = \left[C'\frac{k'_{z2}c}{\omega'\varepsilon_2}\exp(ik'_{z1}d_1)\right]\exp(-ik'_{z2}z), \quad d_1 \leq z < d. \tag{27}$$

One can derive analogous relations for the second-harmonic field in the case $k'_{z2}d_2 = l\pi$. Specifically, to obtain expression for E_{x1}, E_{x2} one should exchange the indices 1 and 2 in Eq. (27); the values ε_2 and ε_{xx} should be exchanged also. If Bragg resonance conditions are satisfied for one layer inside the unit cell [Eq. (24b) or (24c)], the expressions for the field components (27) imply that a standing wave exists in one layer (say, slab 1), while a traveling wave propagates inside another slab (say, slab 2). Thus the periodic structure can be represented as a chain of coupled resonators distributed in space. The existence of a traveling wave means a complete matching between the resonators.

V. PROPERTIES OF THE COEFFICIENT OF NONLINEAR INTERACTION

The complete expression for $W_{k,k',k''}$ [Eq. (18)] is unwieldy and we do not represent it here. Nevertheless, the form of this coefficient is simple and can be discussed. Each term in the integral (18) results in four multipliers such as

$$\frac{\cos k_s d_1 - 1 + i\sin k_s d_1}{k_s d_1}, \tag{28}$$

where k_s stands for one of the combinations

$$k_{z1} + k'_{z1} + k''_{z1}, \tag{29a}$$

$$k_{z1} - k'_{z1} + k''_{z1}, \tag{29b}$$

$$k_{z1} + k'_{z1} - k''_{z1}, \tag{29c}$$

$$k_{z1} - k'_{z1} - k''_{z1}. \tag{29d}$$

For the TWI phenomena in a homogeneous medium, the phase matching conditions are fulfilled for the transversal wave-number components k_z, k'_z, k''_z. Therefore, one of the sums in Eq. (29) will be zero and the multiplier (28) is imaginary. Therefore, in a homogeneous medium, $W_{k,k',k''}$ is imaginary also. In a periodic structure, the law of synchronism is completed for the Bloch wave numbers $\overline{k}, \overline{k'}, \overline{k''}$ [Eq. (17c)]. Thus, in general, none of relations (29) is zero. Hence the multiplier (28) is complex. The latter results in complex $W_{k,k',k''}$. Nonzero real and imaginary parts of $W_{k,k',k''}$ result from the fact that the fields of the three waves accumulate different phase shifts along the path between two slab interfaces. One can say that a ''mismatch'' between the optical widths of the layers takes place.

If one of the sums (29) is zero, the corresponding value of the multiplier (28) is imaginary and has a maximum absolute value. In our problem, the relation $k_{z1} - k'_{z1} - k''_{z1} = 0$ [see Eq. (29d)] is identical because $k'_{z1} \equiv k''_{z1}$ and $k_{z1} = 2k'_{z1}$ in a dispersionless medium [see the paragraph after Eqs. (21)]. Here and in what follows we shall consider in Eq. (18) only one term containing $k_s = k_{z1} - k'_{z1} - k''_{z1} \equiv 0$.

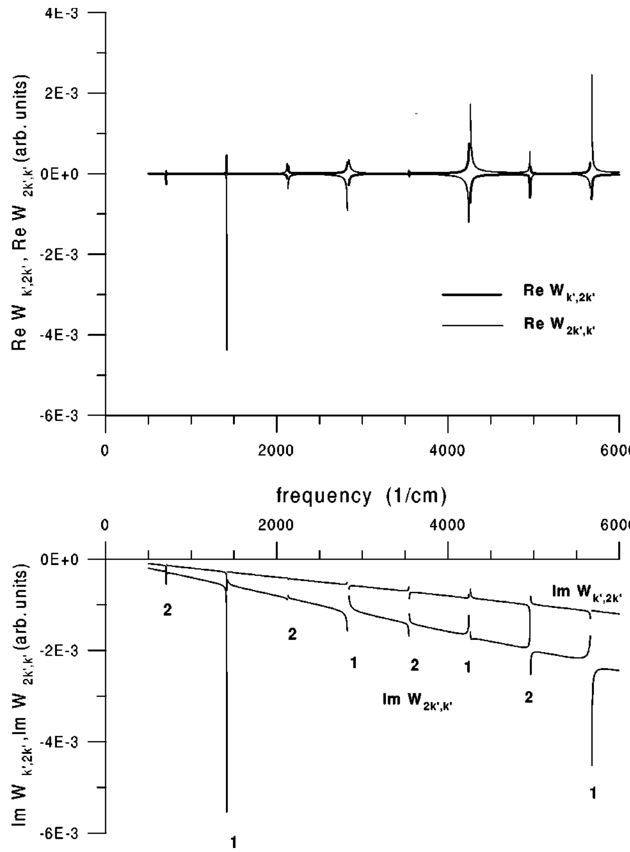


FIG. 2. Dependence of the nonlinear interaction coefficients on frequency for the case of Eq. (24a): real (top) and imaginary (bottom) parts of $W_{k',2k'}$ and $W_{2k',k'}$. The breaks in the curves are associated with the stop bands in Fig. 1: 1, stop bands of the first-harmonic wave; 2, stop bands of the second-harmonic wave.

It should be noted that the discussion above is not valid if the parameters of waves, namely, either ω, k_x or ω', k'_x, are close to the passband edges. In the vicinity of the passband boundary, the nonlinear elements $W_{2k',k'}$ and $W_{k',2k'}$ [Eq. (20)] have singularities, as it will be shown below in Figs. 2 and 3.

Figure 2 shows the real and imaginary parts of the nonlinear coefficients as functions of frequency for the case Eq. (24a). It is seen in the figure that the curves have breaks near the passband edges where $\overline{k'd} = 0$ or π (see Fig. 1). The breaks marked by 1 are associated with the stop bands of the first-harmonic wave and label 2 stands for the band gap of the second-harmonic wave. The Bloch wave number of the first harmonic equals $\pi/2$. It is seen in Fig. 2 that in the vicinity of the stop bands, the real and imaginary parts of the nonlinear coefficients vary considerably. Specifically, the absolute value of the real part increases from zero to values comparable to those absolute values of the imaginary part. It turns out that $\mathrm{Re}(W_{2k',k'})\mathrm{Re}(W_{k',2k'}) < 0$ and both $\mathrm{Im}(W_{2k',k'}) < 0$ and $\mathrm{Im}(W_{k',2k'}) < 0$. Also, $W_{k',2k'} = W_{2k',k'}/2$.

A physical reason for such considerable varying (or

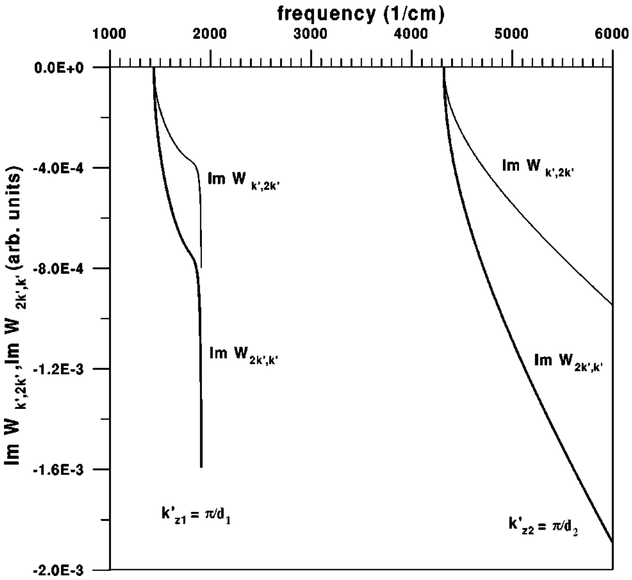


FIG. 3. Imaginary part of the nonlinear interaction coefficients as a function of frequency for Eq. (24b) (left-hand curves) and Eq. (24c) (right-hand curves). Re $W_{k',2k'}=\text{Re } W_{2k',k'}=0$.

breaks) of the nonlinear coefficients is as follows. There is a point inside a stop band where $m_{12}=0$ [23] that is associated with the Bragg resonance of the whole structure period. Indeed, if the conditions (24a) are satisfied, then

$$m_{12}=-i\frac{\omega}{c}\frac{\varepsilon_2}{k_{z2}}\sin(k_{z1}d_1+k_{z2}d_2)=0.$$

That is, $k_{z1}d_1+k_{z2}d_2=n\pi$ is the Bragg resonance condition. As it follows from Eq. (9), if $m_{12}=0$, the coefficients A,B_1,B_2 tend to infinity. As consequence, $W_{k',2k'}$ and $W_{2k',k'}$ diverge. In the vicinity of the passband edge, $W_{k',2k'}$ and $W_{2k',k'}$ are finite values that depend on the detuning between the frequency of the band edges and the frequency that gives $m_{12}=0$. In other words, the nonlinear coefficients increase when the wave frequency approaches the frequency of the Bragg resonance of the lattice period.

Figure 3 shows imaginary parts of the nonlinear coefficients for the case when Bragg resonance conditions are fulfilled for one single layer inside the unit cell, namely, Eq. (24b) or (24c). Now Re $W_{k',2k'}=0$ and Re $W_{k',2k'}=0$. One can see in the figure that the absolute values of the nonlinear coefficients increase when the frequency approaches the band edges. A physical explanation is as above, namely, the Bragg resonance conditions are satisfied not only inside each of the layers inside the unit cell but for the whole period as well. In contrast to Fig. 2, this kind of resonance occurs at the passband edges. Therefore, in order to obtain the coefficients A,B_1,B_2 one should pass to the limit in Eq. (9):

$$A=\lim_{k_{z1}d_1+k_{z2}d_2\to n\pi}\frac{m_{22}-e^{i\bar{k}d}}{m_{12}}.$$

Now one can verify that the nonlinear coefficients are finite values.

If the Bragg resonance conditions are satisfied for one layer inside the unit cell, the following analytic expression for the nonlinear coefficients can be derived:

$$W_{k',2k'}=-i\frac{8\pi c}{\omega'}\frac{\varepsilon_{xx}}{\varepsilon_{zz}}\frac{k'_xd_1}{k'_{z1}}\left[(\chi_{xxz}+\chi_{zxx})\frac{k'^2_{z1}}{\varepsilon^2_{xx}}+\chi_{zzz}\frac{k'^2_x}{\varepsilon^2_{zz}}\right],$$

$$W_{2k',k'}=2W_{k',2k'}. \tag{30}$$

Here we suppose $k'_{z2}=\pi/d_2$.

Equations (30) imply that the nonlinear coefficient is proportional to the relative width of the nonlinear layer. The anisotropy of the dielectric permittivity has a weak influence on $W_{k',2k'}$ because $\varepsilon_{xx}\approx\varepsilon_{zz}$ for the optical frequency. Nevertheless, depending on the relation between the wave-vector components k'_x and k'_{z1}, the anisotropy considerably affects the nonlinear properties. When $k'_x\to 0$ (see Fig. 1), the susceptibility components χ_{xxz} and χ_{zxx} have the most influence on $W_{k',2k'}$. With the increase of k'_x, k'_{z1} decreases and the influence of χ_{zzz} dominates. It should be noted that the studies of Refs. [2,3,5] use a method that determines only the dependence of the nonlinear coefficient on one component of the nonlinear susceptibility, namely, on the χ_{xxz} component. Reference [1] has stated the increase of the nonlinear interaction coefficient at the passband edges.

VI. ANALYSIS OF COUPLING EQUATIONS

To solve Eqs. (20) we introduce the notation

$$C=\bar{C}e^{i\varphi},\quad C'=\bar{C}'e^{i\varphi'},\quad W_{2k',k'}=We^{i\theta},$$

$$W_{k',2k'}=W'e^{i\theta'},\quad \Phi=\varphi-2\varphi'. \tag{31}$$

Then the coupling equations (20) transform into

$$\frac{d\bar{C}}{dz}=W\bar{C}'^2\cos(\theta+\Phi),$$

$$\frac{d\bar{C}'}{\text{dz}}=W'\overline{CC'}\cos(\theta'-\Phi), \tag{33}$$

$$\frac{d\Phi}{\text{dz}}=-W\frac{\bar{C}'^2}{C_k}\sin(\theta+\Phi)+2W'\bar{C}\sin(\theta'-\Phi).$$

If $\theta\neq\theta'$, then only a numerical integration of Eq. (32) is possible [16].

In the cases of Bragg resonances (24b) or (24c), as it follows from Eq. (30) and Fig. 3, the nonlinear elements $W_{k',2k'}$ and $W_{2k',k'}$ are imaginary, i.e., $\theta=\theta'=3/2\pi$. Therefore, Eq. (32) has two first integrals [14]

$$K_1=\frac{\bar{C}^2}{W}+\frac{\bar{C}'^2}{W'},\quad K_2=\overline{CC'}^2\cos\Phi, \tag{33}$$

where K_1 and K_2 are the constants determined by the initial conditions at $z=z_0$. Now the system of equations (32) converts into an elliptic integral and possesses a solution in the form of elliptic functions

$$\overline{C}^2 = WK_1\left\{ y_1 + (y_2 - y_1) \times \mathrm{sn}^2\left[W' W^{1/2} K_1^{1/2}(z-z_0), \sqrt{\frac{y_2-y_1}{y_3-y_1}}\right]\right\},$$

where $y_1 < y_2 < y_3$ are the roots of the cubic equation

$$y(1-y)^2 - K = 0, \quad K = \frac{K_2^2}{K_1^3 W (W')^2}.$$

If the amplitude of the second-harmonic wave is small at the coordinate origin, namely, $\overline{C}(0) \ll \overline{C}'(0)$, then

$$K_1 \simeq \frac{\overline{C}'(0)^2}{W'}, \quad K_2 \simeq \overline{C}(0)\overline{C}'(0)^2 \cos\Phi(0),$$

$$K \simeq \left(\frac{\overline{C}(0)}{\overline{C}'(0)}\right)^2 \frac{W'}{W}\cos^2\Phi(0).$$

$K \leqslant 1$ and K is independent of W and W' because $W'/W = 1/2$ always. Now one can approximately derive the roots of the cubic equation: $y_1 \approx K$ and $y_{2,3} \approx 1 \pm \sqrt{K}$. Consequently, the amplitude of the first and second harmonics can be expressed as

$$\overline{C}^2 \approx \frac{W}{W'}[\overline{C}'(0)]^2 \mathrm{sn}^2[\sqrt{WW'}\,\overline{C}'(0)z, 1-\sqrt{K}],$$
$$\overline{C}'^2 \approx [\overline{C}'(0)]^2\{1 - \mathrm{sn}^2[\sqrt{WW'}\,\overline{C}'(0)z, 1-\sqrt{K}]\}. \quad (34)$$

The existence of the first integrals (33) means that the total energy of the kth and k'th harmonic waves is conserved along the Z axis. By analyzing the mutual variations of $\overline{C}$ and $\overline{C}'$ one can study how these harmonic waves exchange their energy with each other.

In Fig. 4(a) the amplitude $\overline{C}$ of the second-harmonic wave increases almost from zero up to a maximum value at a distance of about 25 structure periods, while $\overline{C}'$ decreases from a maximum to a minimum at the same distance. Thus the energy of the fundamental wave (the k'th wave) converts into the energy of the second-harmonic wave (the kth wave). All curves in Fig. 4(a) have been calculated at the frequency $\omega_A = 1700\ \mathrm{cm}^{-1}$. This point A lies on the curve $k'_{z1} = \pi/d_1$ far from the band edge. In Fig. 4(b) we present an analogous calculation for $\overline{C}$, but at the frequency $\omega_B = 1906\ \mathrm{cm}^{-1}$. A distance that corresponds to the amplitude increase from zero up to a maximum value equals 12 structure periods. A comparison shows that the imaginary parts of W and W' associated with point B are almost 2 times greater than the corresponding values at point A: $\mathrm{Im}\,W = -1.4\times10^{-3}$, $\mathrm{Im}\,W' = -7.0\times10^{-4}$ and $\mathrm{Im}\,W = -6.7\times10^{-4}$, $\mathrm{Im}\,W' = -3.3\times10^{-4}$, respectively. As it follows from Eq. (34), if both the frequency ω' and wave number k'_x increase in such a way that the Bragg resonance condition is satisfied (e.g., one moves from point A toward B along the curve $k'_{z1} = \pi/d$ in Fig. 1), the period of the variation of the amplitudes $\overline{C}$ and $\overline{C}'$ decreases considerably. The latter is due to the increase of the argument of the elliptic function near the passband boundary. Therefore, the periodicity of the structure results in the following: The distance of the energy exchange between the harmonic waves becomes a minimum in the vicinity of the passband edges.

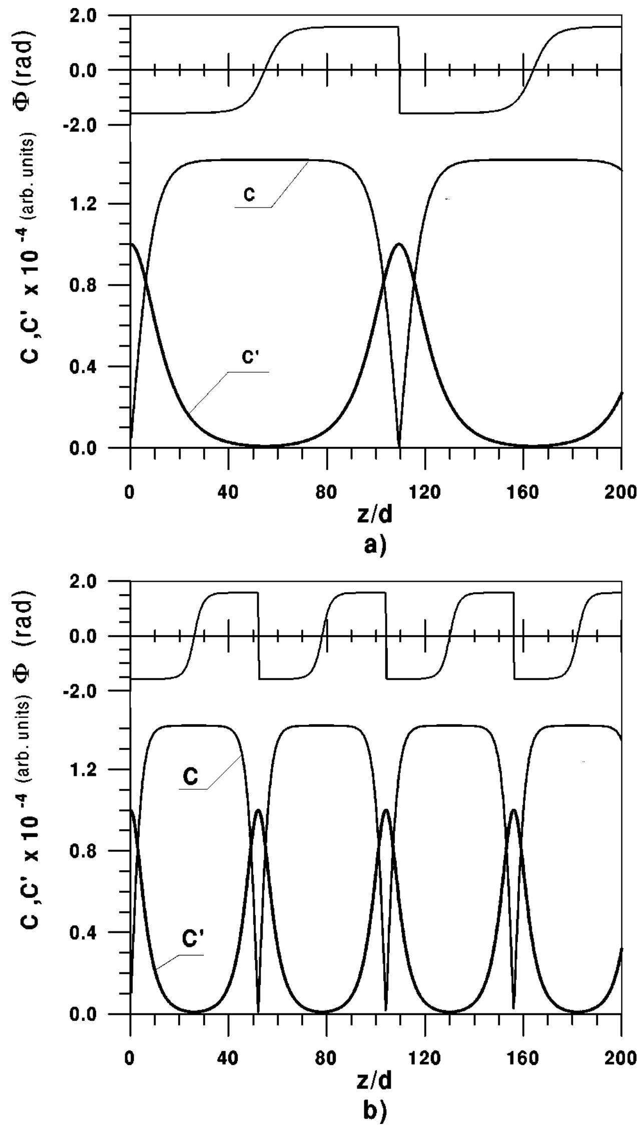
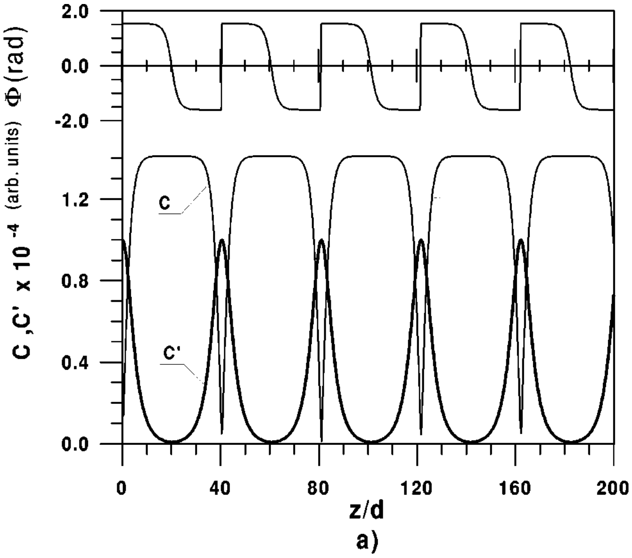


FIG. 4. Dependence of absolute values $\overline{C}, \overline{C}'$ and phase Φ of nonlinear waves on the dimensionless coordinate z/d for the case of Eq. (24b): (a) $\omega = 1700\ \mathrm{cm}^{-1}$, $\mathrm{Im}\,W = -6.7\times10^{-4}$, and $\mathrm{Im}\,W' = -3.3\times10^{-4}$ and (b) $\omega = 1906\ \mathrm{cm}^{-1}$, $\mathrm{Im}\,W = -1.4\times10^{-3}$, and $\mathrm{Im}\ W' = -7.0\times10^{-4}$.

If the phase matching condition (24a) is fulfilled, $\theta \approx \theta'$. Let us make the following variable changes: $\theta = 3/2\pi + \vartheta$ and $\theta' = 3/2\pi - \vartheta'$. If $\vartheta = \vartheta'$, on substituting $\Phi = 2\varphi' - \varphi + 3/2\pi$ into Eq. (32), one obtains an integrable system again, but Φ varies within the limits that depend on ϑ.

In the vicinity of the passbands, both $W_{2k',k'}$ and $W_{k',2k'}$ diverge (see Fig. 2). As a consequence, $\vartheta \neq \vartheta'$; however, the difference between ϑ and ϑ' is small. In this case, Eq. (32) may be solved only numerically. Figure 5 shows a numerical solution of Eq. (32) for $\vartheta = 1.2741$ and $\vartheta' = 1.2723$. In contrast to the behavior of $\overline{C}(z)$ and $\overline{C}'(z)$ in Fig. 4, the har-



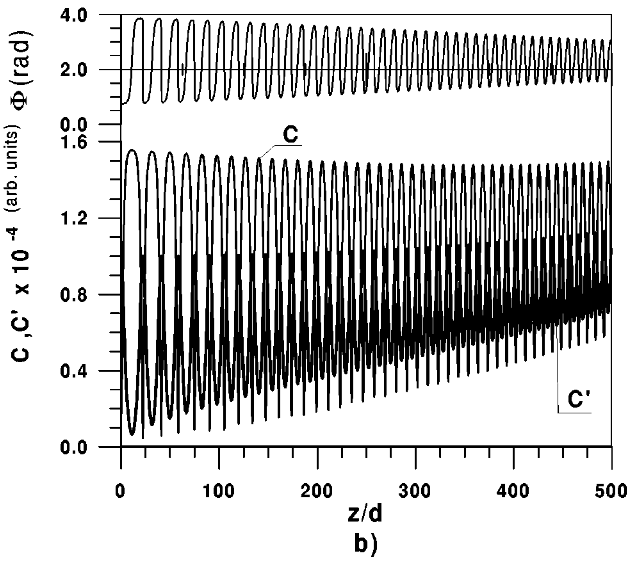


FIG. 5. Same as in Fig. 4 but for Eq. (24a): (a) $\omega = 4200\ \mathrm{cm}^{-1}$, $\mathrm{Im}\,W = -5.5\times10^{-4} - i1.8\times10^{-3}$, and $\mathrm{Im}\,W' = -2.8\times10^{-4} - i9.1\times10^{-4}$ and (b) $\omega = 4260\ \mathrm{cm}^{-1}$, $\mathrm{Im}\,W = -1.7\times10^{-3} - i1.6\times10^{-3}$, and $\mathrm{Im}\,W' = -7.2\times10^{-4} - i6.5\times10^{-4}$.

monic amplitudes in Fig. 5 are not periodic functions of z: $\overline{C}(z)$ and $\overline{C'}(z)$ oscillate along z, but the period of these oscillations also varies slowly along the Z axis and the minimum values of $\overline{C}$ and $\overline{C'}$ increase. If the wave path along z exceeds a certain distance, both the period of the variation of the harmonic amplitudes and the minimum values of $\overline{C}$ and $\overline{C'}$ become independent of z. This distance depends on the mismatch between $\mathrm{Re}\,W_{2k',k'}$ and $\mathrm{Re}\,W_{k',2k'}$ [i.e., on value of $|\theta-\theta'|$; see Eq. (32)]. For instance, in Fig. 5(a), such a distance is about 40 structure periods. In contrast, for the frequency near the passband edge [Fig. 5(b)], both minimum values of $\overline{C},\overline{C'}$ and the period of the variation of the harmonic amplitudes still depend on z even though the wave path exceeds several hundreds of periods.

VII. CONCLUSION

The main result of this work is the analysis of the solution of the three-wave interaction problem inside a structure with artificial symmetry. Here we present a technique that reduces the problem to a known system of coupling equations [14]. Our method is based on Green's formula (15) and allows one to derive coupling equations for an arbitrary functional dependence of the nonlinear dielectric permittivity on the electric field amplitude. This means that not only a well-known Kerr-like nonlinearity (such as $\alpha|E|^2$) but any other dependence can be considered (cf. $\chi_{ijk}E_jE_k, \{j,k\}=x,y,z$). We assume that the nonlinearity is weak, namely, the nonlinear terms in Maxwell equations are small. For nonlinear optical materials of interest to us [see the paragraph after Eq. (1)] this assumption is surely satisfied. Our method allows one to study both the nonlinear processes inside the slabs and those taking place at the structure interfaces.

It is shown that the translation symmetry of the structure changes the conservation laws for the transversal wave-vector component (i.e., Bloch wave vector). In general, the coefficient of the nonlinear interaction is complex even without dissipation. In contrast, for the TWI processes in a homogeneous medium, the coefficient of nonlinear interaction is imaginary.

It is stated that, during the generation of the second-harmonic wave, the laws of energy and pulse conservation [Eqs. (17a) and Eqs. (17b) and (17c), respectively] of the interacting waves are associated with the Bragg resonances of either a single layer or the whole period of the structure. The latter type of Bragg resonance implies that the conservation laws are completed for the structure with an effective dielectric permittivity given by Eq. (25). For this case, the phase matching conditions imply that a traveling wave propagates in the periodic structure.

The particularities of the nonlinear interaction associated with the complex character of the matrix elements $W_{2k',k'}$ and $W_{k',2k'}$ have been studied. The increase of the coefficient of the nonlinear interaction is associated with the Bragg resonant conditions. During the Bragg resonance, one observes a considerable decrease of the wave path associated with the complete energy exchange between the harmonics propagating inside a lossless periodic structure.

From our point of view, the present study has both a basic interest and practical applications for the purposes of spectroscopy and for the determination of periodic structure parameters. Specifically, Fig. 3 allows a measurement of the stop band parameters. Therefore, the configuration and dielectric characteristics of layers may be determined. As it was mentioned, for a periodic structure, the distance of the complete energy transfer between the harmonics may be shortened significantly. Thus periodic structures may be effectively used in amplifiers and mixers of optical signals.

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[21] Solving the nonlinear problem, we assume that the slowly varying amplitude C of the kth wave depends only on z. Strictly speaking, C is a function of both z and x. Nevertheless, if one takes into account the complete dependence $C(z,x)$, the coefficient of the nonlinear interaction $W_{k,k',k''}$, first introduced in Eq. (18) and discussed in Sec. V, will be changed only quantitatively. However, it affects neither the phase matching conditions nor the dynamics of the nonlinear process. Thus the results obtained in this article are qualitatively correct.
[22] In the linear operator, apart from the terms with $\vec{e}$ and h_y, the additional fields $\vec{e}^{(ad)}$ and $h_y^{(ad)}$ exist. The corresponding integrals will be zero if we assume that the additional fields satisfy the same boundary conditions as the main field components.
[23] The fact that $m_{12}=0$ at the band-gap frequency may be demonstrated as follows (see Ref. [8]). The matrix $\hat{m}$ in Eq. (4) relates the fields of two equivalent layers with the same index of refraction; thus $\hat{m}$ is unimodular, i.e., $m_{11}m_{22}-m_{12}m_{21}=1$ (cf. Ref. [7]). Then $m_{11}m_{22}=1$ and from Eq. (7) one has $|\cos \overline{k}d|=|m_{11}+1/m_{11}|/2>1$ $(|m_{11}|\neq 1)$ and thus $\overline{k}$ is imaginary.